

Realization of Discrete Quantum Billiard in 2D Optical Lattices

Dmitry O. Krimer^{1,2*} and Ramaz Khomeriki^{2,3†}

¹*Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, 72076 Tübingen, Germany*

²*Max-Planck Institute for the Physics of Complex Systems, Nöthnitzer Str. 38, 01187 Dresden, Germany*

³*Physics Department, Tbilisi State University, Chavchavadze 3, 0128 Tbilisi, Georgia*

We propose the method for optical visualization of Bose-Hubbard model with two interacting bosons in the form of two-dimensional (2D) optical lattices consisting of optical waveguides, where the waveguides at the diagonal are characterized by different refractive index than others elsewhere, modeling the boson-boson interaction. We study the light intensity distribution function averaged over direction of propagation for both ordered and disordered cases, exploring sensitivity of the averaged picture with respect to the beam injection position. For our finite systems the resulting patterns reminiscent the ones set in billiards and therefore we introduce a definition of discrete quantum billiard discussing the possible relevance to its well established continuous counterpart.

PACS numbers: 67.85.-d, 37.10.Jk, 03.65.Ge

A very rich variety of wave phenomena that have originally been discovered in the context of atomic and solid state physics attracted recently much attention due to their deep analogy to optical systems. A first prominent example is the Anderson localization, the phenomenon which was originally discovered as the localization of electronic wavefunction in disordered crystals [1] and later understood as a fundamental concept being universal phenomenon of wave physics. Related recent experiments were performed on light propagation in spatially random nonlinear optical media [2, 3] and on Bose-Einstein condensate expansions in random optical potentials [4]. A second example is the well known solid state problem of an electron in a periodic potential with an additional electric field, which lead to investigations of Bloch oscillations and Landau-Zener tunneling in various physical systems such as ultracold atoms in optical lattices [5–7] and optical waves in photonic lattices [8, 9]. A recent progress in the experiments stimulated a new turn in theoretical studies dealing with the evolution of a wave packet in nonlinear disordered chains [10], in a nonlinear Stark ladder [11] and the effect of Anderson localization of light near boundaries of disordered photonic lattices [12] which are just a few recent examples. A third very interesting example is a classical analog of beam dynamics in one-dimensional (1D) photonic lattices to quantum coherent and displaced Fock states [13] and a classical realization of the two-site Bose-Hubbard model (applicable to the physics of strongly interacting many-body systems), based on light transport in engineered optical waveguide lattices [14].

In this Letter we study a classical analog of beam propagating in 2D photonic lattices to quantum coherent dynamics of two particles in one-dimensional chain using the Bose-Hubbard model. We consider different situations ranging from the simple ordered case without inter-

action to the disordered case with interaction in our finite systems. Sometimes the resulting patterns look pretty similar as the ones for the classical and/or quantum billiards which are known to exhibit regular and chaotic behaviors (see e.g. Ref. [15]). We would like to emphasize particularly the growing interest to the two-particle problem in the context of quantum correlations between two noninteracting particles evolving simultaneously in a disordered medium [16] and quantum walks of correlated photons which provides a route to universal quan-

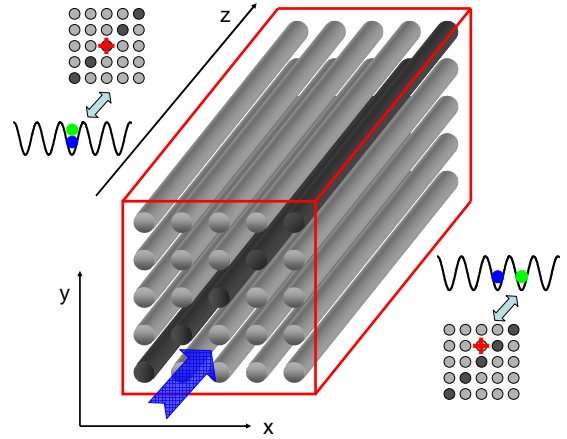


FIG. 1: Geometry of setup: A beam enters into 2D optical lattice, and propagates along the z axis. The refractivity index is invariant along the z axis and either periodic or disordered in transverse directions. The corresponding mapping to the dynamics of two interacting distinguishable bosons in a chain is also done (see the text for details). The interaction between bosons is introduced by taking the refractive index for the diagonal waveguides different from the rest. The injection of a beam to the diagonal waveguide mimics launching initially both bosons at the same site (upper inset), while injecting the beam into off-diagonal waveguide corresponds to the two bosons located initially on different sites (lower inset).

*Electronic address: dmitry.krimer@gmail.com

†Electronic address: khomeriki@hotmail.com

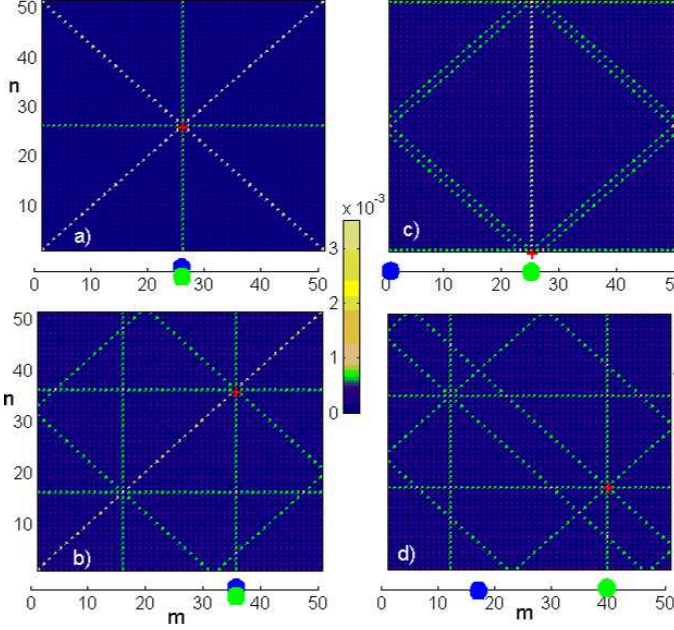


FIG. 2: (Color online) Characteristic pictures for discrete quantum billiard realization for different injection points depicted by a red cross in the absence of disorder and interaction ($W = 0$ and $U = 0$). In the main graphs the averaged PDFs [see Eq. (3)] are displayed and accompanied by the lower insets which show the numbers of initially injected waveguides (or number of sites at which the particles are initially located).

tum computation [17]. Thus, the obtained results might be applicable to both classical and quantum systems.

Let us introduce a standard Bose-Hubbard Hamiltonian describing two distinguishable bosons (or two fermions with opposite spins) in a chain with N sites

$$\hat{\mathcal{H}} = \sum_{j=1}^N \left[\left(\hat{a}_{j+1}^\dagger \hat{a}_j + \hat{b}_{j+1}^\dagger \hat{b}_j + h.c. \right) + U \hat{a}_j^\dagger \hat{a}_j \hat{b}_j^\dagger \hat{b}_j \right] \quad (1)$$

where \hat{a}_j^\dagger (\hat{a}_j) and \hat{b}_j^\dagger (\hat{b}_j) are bosons creation and annihilation operators on a lattice site j and U is the on-site interaction strength. Starting from the time dependent Schrödinger equation $i\partial_t|\Psi(t)\rangle = \hat{\mathcal{H}}|\Psi(t)\rangle$ we expand $|\Psi(t)\rangle$ in terms of the N^2 orthonormal eigenstates of a number operator, $|m, n\rangle \equiv \hat{b}_m^\dagger \hat{a}_n^\dagger |0\rangle$, as $|\Psi(t)\rangle = \sum_{m,n=1}^N c_{mn}(t) |m, n\rangle$, where the amplitudes $c_{mn}(t)$ satisfy to the following set of equations

$$i\dot{c}_{mn} = U\delta_{mn}c_{mn} + \sum_{m',n'=1}^N R_{mn}^{m'n'} c_{m'n'}, \quad R_{mn}^{m'n'} = \quad (2)$$

$$\delta_{m'/m+1}\delta_{n'/n} + \delta_{m',m-1}\delta_{n'/n} + \delta_{m'/m}\delta_{n'/n+1} + \delta_{m'/m}\delta_{n'/n-1}.$$

Note that Eq. (2) is invariant under permutation of m and n , and therefore it is natural to represent c_{mn} as a sum of symmetric $c_{mn}^S = (c_{mn} + c_{nm})/\sqrt{2}$ and anti-symmetric $c_{mn}^A = (c_{mn} - c_{nm})/\sqrt{2}$ functions. In such

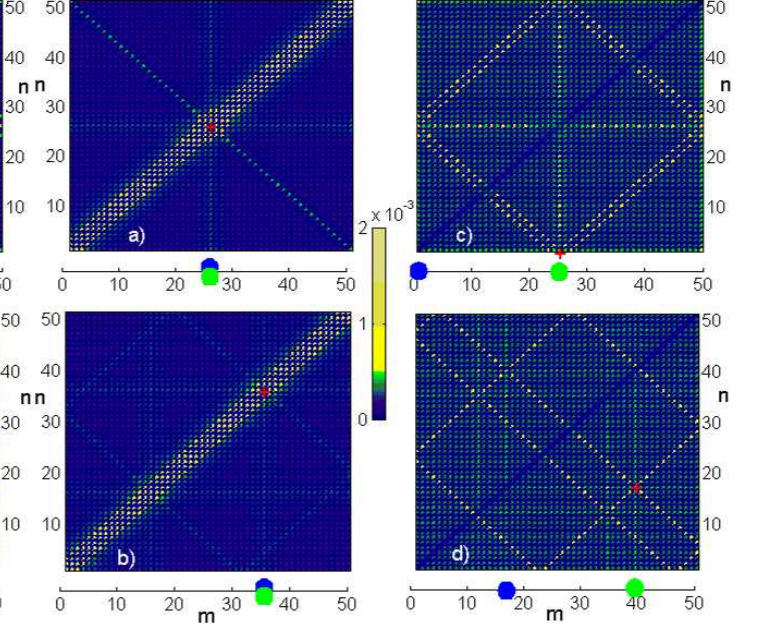


FIG. 3: (Color online) All parameters and quantities are the same as in Fig. 2, except the interaction constant $U = 1$.

a basis the matrix $R_{mn}^{m'n'}$ is decomposed into two irreducible parts, one of which corresponds to the Bose-Hubbard model with two indistinguishable bosons and the other describes the physics of two indistinguishable spinless fermions. For the symmetric initial conditions, $c_{mn}(0) = c_{nm}(0)$, the dynamics is reduced to the former case (two indistinguishable bosons on sites m and n), whereas the latter case is realized for the antisymmetric initial conditions, $c_{mn}(0) = -c_{nm}(0)$.

Remarkably, Eq. (2) is the same as one used for the description of light propagation through 2D optical lattices (see Fig. 1) within the tight-binding approximation, where longitudinal dimension z plays a role of time. This approximation is valid when a lattice is constructed such that tunneling into nearest neighboring waveguides is allowed only and there is a difference between the refractive indices of the diagonal n_d and off-diagonal n_0 waveguides which models the interaction (with the interaction strength $U \sim n_0 - n_d$). Thus, injecting light beam at the waveguide with a position $x = m, y = n$ (asymmetric initial conditions) corresponds to the dynamics of two distinguishable interacting bosons in a chain, placed initially on sites m and n . One can also think about the Bose-Einstein condensate embedded into 2D optical lattice and then Eq. (2) describes the evolution of some initial matter wave packet through the lattice.

In this Letter we consider the system with hard boundaries having $c_{mn} = 0$ outside a square and monitor the time averaged wave function

$$P_{mn} \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |c_{mn}(t)|^2 dt, \quad (3)$$

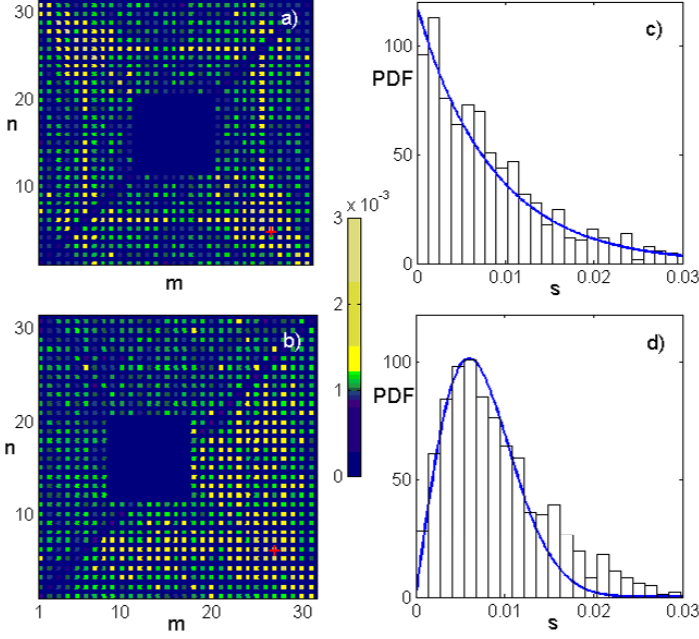


FIG. 4: (Color online) Characteristic pictures for discrete quantum billiard realization with a rigid square placed inside a system for the interaction constant $U = 1$ and the same injection point depicted by a red cross. (a),(b): symmetric and asymmetric situations, respectively, with the corresponding probability density functions of eigenvalue spacings s shown in (c) and (d). (c): the Poisson distribution (6), with the average spacing $d = 0.0085$. (d): the Wigner-Dyson distribution (7), with the average spacing $d = 0.0075$.

referring to P_{mn} as to the averaged two-particle probability distribution function (PDF). To calculate PDFs we, at first, solve the eigenvalue problem $\hat{H}|q\rangle = \lambda_q|q\rangle$ and then expand $c_{mn}(t)$ with respect to the eigenvectors as

$$c_{mn}(t) = \sum_{q=1}^{N^2} \phi_q \mathcal{L}_{mn}^{(q)} e^{-i\lambda_q t}, \quad (4)$$

where $\mathcal{L}_{mn}^{(q)} \equiv \langle q|m, n\rangle$ is the eigenvector which belongs to the eigenvalue λ_q and $\phi_q \equiv \sum_{m,n=1}^N c_{mn}(0) \mathcal{L}_{mn}^{(q)}$ is its initial amplitude. Next, the averaged PDF is calculated by the following formula:

$$P_{mn} = \sum_q |\varphi_q|^2 \mathcal{L}_{mn}^{(q)2} + \sum_i \left| \sum_{q_i^r} \varphi_{q_i^r} \mathcal{L}_{mn}^{(q_i^r)} \right|^2, \quad (5)$$

where the first sum runs over all nondegenerate eigenvalues, whereas the second sum corresponds to the summation with respect to r -fold degenerate eigenvalues λ_{q_i} .

Intuitively it seems that the light injected into one of the waveguides should spread over a whole lattice, however the real situation is completely opposite due to the interference from the hard boundaries. Let us start from

the simplest noninteracting case, $U = 0$ (for the optical counterpart shown in Fig. 1 waveguides must all be identical). As is seen from Fig. 2 a well defined pattern for P_{mn} corresponds to each initial injection point. Thus, the system keeps the information about its initial state and from the averaged picture one can recover an initial signal. It should be noted, that these patterns might be strongly modified when the interaction is switched on, $U \neq 0$ (see Fig. 3). Remarkably, the pattern's structure reminiscent the one sets in billiards, therefore, we introduce the notion of discrete quantum billiard and seek for the analogies with usual continuous counterparts. The first step towards this direction is to explore the possibility of quantum chaos realization in such systems. We consider two possibilities to observe the transition towards quantum chaos. The first one is a symmetry breaking by placing a square with rigid boundaries inside the system as is shown in Fig. 4. We monitor then the statistical properties of the eigenvalue spacings $s = |\lambda_{q+1} - \lambda_q|$ for different locations of the square, keeping injection point and interaction constant the same. It is seen that in the symmetric case the Poisson distribution

$$P(s) = 1/d \cdot e^{-s/d} \quad (6)$$

is realized, while for the asymmetric case the Wigner-Dyson distribution is observed

$$P(s) = \pi s / (2d^2) \cdot e^{-\pi s^2 / (4d^2)}. \quad (7)$$

Thus the onset of quantum chaos could be visualized via the classical optical system of coupled waveguides.

The second mechanism of quantum chaos realization is an introduction of the disorder via adding the following terms to the Bose-Hubbard Hamiltonian (1)

$$\hat{H}_d = \sum_{j=1}^N \left[\epsilon_j^a \hat{a}_j^\dagger \hat{a}_j + \epsilon_j^b \hat{b}_j^\dagger \hat{b}_j \right], \quad (8)$$

where for the sake of simplicity we take symmetric disorder $\epsilon_j^a = \epsilon_j^b \equiv \epsilon_j$ (ϵ_j are random numbers from the interval $[-W/2, W/2]$, W being the disorder strength). In the optical context presented in Fig. 1 it implies the usage of symmetric (under permutation of m and n) disordered lattice described by the following modified evolution equations:

$$i\dot{c}_{mn} = (W_{mn} + U\delta_{mn}) c_{mn} + \sum_{m',n'=1}^N R_{mn}^{m'n'} c_{m'n'}, \quad (9)$$

where the matrix $R_{mn}^{m'n'}$ is the same as in Eq. (2) and $W_{mn} = \epsilon_m + \epsilon_n$ are correlated disorder parameters. An uncorrelated disorder can also be taken with W_{mn} being a sum of two independent random numbers for each m and n . In the optical context such a situation might be realized by taking either correlated or uncorrelated random refractive index distribution in a whole lattice. Note, that only the former case coincides with the dynamics of two bosons in a disordered chain with a random potential.

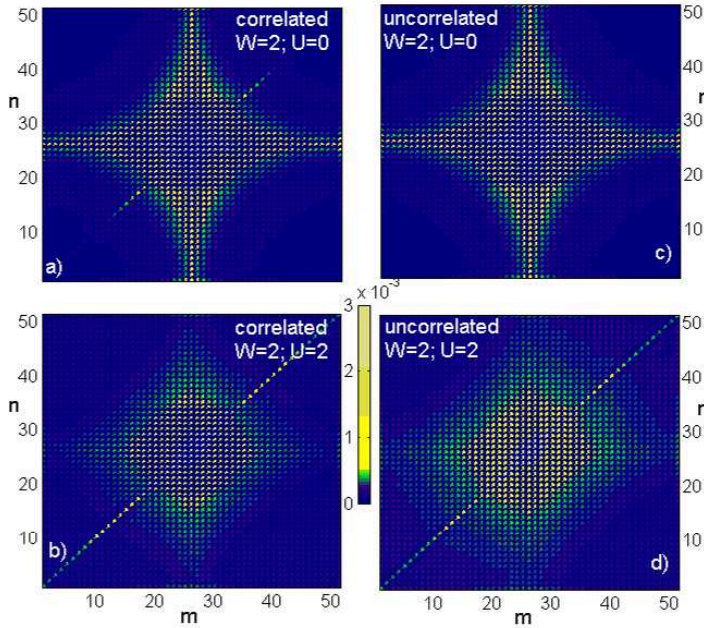


FIG. 5: (Color online) Averaged PDFs both in time and over many disorder realizations (all parameters are shown on the figures). Injection point in all graphs is taken at the middle of 2D optical lattice.

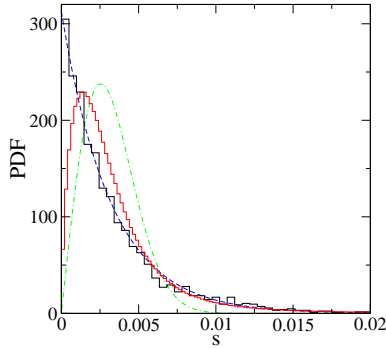


FIG. 6: (Color online) Probability density functions of eigenvalue spacings s for three different cases and a chain with $N = 51$. Black curve: $W=0$, $U=2$ (disorder strength is zero only). Red curve (dark gray): $W=2$, $U=2$. The case with uncorrelated disorder is considered. Green dash-dotted curve: the Wigner-Dyson distribution (7), with the average spacing $d = 0.0032$. Blue dashed curve: the Poisson distribution (6), with the average spacing $d = 0.0032$.

We consider two particles which are initially launched on the same site at the middle of a chain, $m = n = N/2$ (the optical counterpart corresponds to the beam injection at the central waveguide in 2D optical lattices). The typical structures for P_{mn} , averaged out with respect to many disorder realizations, are shown in Fig. 5. As is seen, the averaged PDFs demonstrate well pronounced patterns, which look differently as compared to the case with a single disorder realization, when PDF has many spots at different locations. For the noninteracting case PDF has an anisotropic structure with two distinct directions, $m = N/2$ and $n = N/2$, along which the particle motion mostly develops in average. For the case with correlated disorder, a slight contribution of two particle states is also visible. For $U = 2$, the interaction is already strong enough, such that the contribution of states corresponding to the breather band becomes essential and the two particles mostly prefer to form a composite state and travel together. Interestingly, the level of system's chaoticity might easily be governed by setting different values of disorder and interaction strengths. To demonstrate this we study the statistical properties of the eigenvalue spacings $s = |\lambda_{q+1} - \lambda_q|$ for three different cases as is shown in Fig. 6 (the events with $s = 0$ due to degeneracy are not counted). It is seen that for nonzero disorder and interaction strengths, probability density function of the spacings s has a tendency to go to the Wigner-Dyson distribution and, as a consequence, a system becomes more chaotic.

Concluding, in this paper we have discussed various interpretation of the optical beam propagation through 2D optical crystal ranging from interacting cold atom dynamics and two particle Anderson localization to the quantum billiard problems connected with transition to quantum chaoticity.

Acknowledgements: The authors are indebted to I. Babushkin, S. Denisov and N. Li for usefull discussions regarding the Billiard issues. R.Kh. is supported by RNSF (Grant No. 09/04) and STCU (Grant No. 5053).

-
- [1] P.W. Anderson, Phys. Rev. **109**, 1492 (1958).
 - [2] T. Schwartz *et al*, Nature **446**, 52 (2007).
 - [3] Y. Lahini *et al*, Phys. Rev. Lett. **100** 013906 (2008).
 - [4] J. Billy *et al*, Nature **453**, 891 (2008); G. Roati *et al*, Nature **453**, 895 (2008).
 - [5] M. Gustavsson *et al*, Phys. Rev. Lett. **100**, 080404 (2008).
 - [6] O. Morsch *et al*, Phys. Rev. Lett. **87**, 140402 (2001); G. Ferrari *et al*, *ibid.* **97**, 060402 (2006).
 - [7] B.P. Anderson, M.A. Kasevich, Science **282**, 1686 (1998); M. Ben Dahan *et al*, Phys. Rev. Lett. **76**, 4508 (1996).
 - [8] T. Pertsch *et al*, Phys. Rev. Lett. **83**, 4752 (1999); R. Sapienza *et al*, *ibid.* **91**, 263902 (2003); H. Trompeter *et al*, *ibid.* **96**, 023901 (2006); F. Dreisow *et al*, *ibid.* **102**,

- 076802 (2009).
- [9] R. Morandotti *et al*, Phys. Rev. Lett. **83**, 4756 (1999).
 - [10] A. S. Pikovsky, D. L. Shepelyansky, Phys. Rev. Lett. **100** 094101 (2008); S. Flach, D.O. Krimer, Ch. Skokos, Phys. Rev. Lett. **102** 024101 (2009); M. Johansson, G. Kopidakis, S. Aubry, Europhys. Lett. **91** 50001 (2010); T. V. Laptyeva *et al*, EPL **91** 30001 (2010); A. Pikovsky, S. Fishman, Phys. Rev. E. **83**, 025201(R) (2011).
 - [11] D.O. Krimer, R. Khomeriki, S. Flach, Phys. Rev. E, **80**, 036201 (2009); A. R. Kolovsky, E. A. Gomez, H. J. Korsch, Phys. Rev. A **81**, 025603 (2010).
 - [12] D.M. Jović *et al*, Phys. Rev. A **83**, 033813 (2011).
 - [13] A. Perez-Leija *et al*, Optics Lett. **35** 2409 (2010).
 - [14] S. Longhi, arXiv:1101.4762.
 - [15] H.-J. Stöckmann, *Quantum Chaos - An Introduction* (Cambridge University Press, Cambridge, UK, 1999).
 - [16] Y. Lahini *et al*, Phys. Rev. Lett. **105**, 163905 (2010).
 - [17] M. Karski *et al*, Science **325**, 174 (2009); A. Peruzzo *et al*, Science **329**, 1500 (2010); Y. Lahini *et al*, arXiv:1105.2273.